*Int. J. Solids Structures*, 1974, 10, pp. 285–292. Pergamon Press. Printed in Gt. Britain.

# INTEGRAL TRANSFORMS OF DIFFERENTIAL EQUATIONS FOR DEFLECTIONS OF ISOTROPIC SANDWICH PLATES

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(*Received* 5 *March* 1973; *revised* 18 *May* 1973)

**Abstract**—Integral transforms of derivatives of functions appearing in the equations for deflections of sandwich plates are presented in terms of transforms of those functions and of appropriate boundary values. The use of the generalized Green's formula enables us to present these transforms in an invariant form. A rectangular plate is analyzed as an example.

## NOTATION

The notation which is used in this paper is partially adopted from Wempner and Baylor[1]. Greek suffixes take the values 1 and 2. A semi-colon (;) is used to denote covariant differentiation while a comma (,) denotes partial differentiation. Prefixes ($n$) are used to distinguish between the upper ($n = 0$) or lower ($n = 1$) facings.

| | |
|---|---|
| $a, b$ | dimensions of a rectangular plate |
| $c^\alpha$ | $= {}_0\lambda\, {}_0p^\alpha - {}_1\lambda\, {}_1p^\alpha$ |
| $d$ | thickness of the core |
| ${}_nd$ | thickness of a facing |
| $e_{(i)}, f_{(i)}$ | constants |
| $k_{(i)}$ | eigenvalues ($i = 1, 2, 3$) |
| $p^3$ | the vertical component of the external force |
| ${}_np^\alpha$ | the "in-plane" components of the external forces |
| $\bar{p}^\alpha$ | $= {}_0p^\alpha - \dfrac{1}{\gamma}\, {}_1p^\alpha$ |
| $s^\alpha$ | components of the transverse shear resultants |
| $w$ | vertical displacement of the plate |
| $A$ | $= \dfrac{LG\bar{\lambda}^2}{2C_2} + \dfrac{1-\eta}{2C_1}$ |
| $B$ | $= (1-\eta)/2C_1C_2L$ |
| $C_1$ | $= 2\lambda\, {}_0\lambda\, {}_0\mu\gamma/G(1+\gamma)$ |
| $C_2$ | $= {}_0\lambda\, {}_0\mu L({}_0\lambda^2 + \gamma {}_1\lambda^2)/6(1-\eta)$ |
| $G$ | transverse shear modulus of an isotropic core |
| $L$ | a characteristic length of the middle surface |
| $P$ | $= L[p^3 + (\frac{1}{2})c^\beta{}_{;\beta} - 2C_1(p^3{}_{;\alpha}{}^{\alpha} + \frac{1}{2}c^\beta{}_{;\beta\alpha}{}^{\alpha})/(1-\eta)$ $+ \bar{\lambda}\gamma\bar{p}^\beta{}_{;\beta}/(1+\gamma)]$ |
| $\gamma$ | $= {}_1\lambda\, {}_1\mu/{}_0\lambda\, {}_0\mu$ |
| $\eta$ | Poisson's ratios of both facings |
| $\lambda$ | $= d/2L$ |
| ${}_n\lambda$ | $= {}_nd/L$ |
| $\bar{\lambda}$ | $= 2\lambda + ({}_0\lambda + {}_1\lambda)/2$ |
| ${}_n\mu$ | shear modulus of a facing |
| $\phi_{(i)}$ | Orthogonal system of eigenfunctions |
| $\theta^\alpha$ | dimensionless surface coordinates. |

## INTRODUCTION

Recently, Wempner and Baylor[1] derived equations for the deflection of sandwich plates with von Kármán type of geometric nonlinearities. When the contraction of the core was neglected, isotropy was assumed, and the nonlinear terms were disregarded, a system of three partial differential equations resulted. It is well known that such a system of equations is well suited for application of some type of integral transform. It is equally well known that the simplicity of application of integral transforms usually is disturbed by a cumbersome procedure of integration by parts. In addition, once one attempts to solve the same problem for an area of different shape, the whole procedure must be repeated because, in general, a different kernel of transformation must be used. In another paper[2] this author showed how to avoid this difficulty in the case of the Helmholtz's equation and in the case of the equation describing the bending of a homogeneous plate. In both cases, Green's theorem served as a tool to achieve a unifying goal. In the present paper this concept is generalized to include a system of equations from[1] (one scalar equation of order six and one vector equation). The use of generalized Green's theorem[3] is applicable in this case since vector equations are involved. An application to a rectangular plate illustrates the procedure developed herein.

## INTEGRAL TRANSFORMS OF EQUATIONS FOR DEFLECTION

Consider equations (39) and (40) from paper[1]

$$w_{;\alpha\beta\gamma}^{;\alpha\beta\gamma} - Aw_{;\alpha\beta}^{;\alpha\beta} + BP = 0, \tag{1}$$

$$s^{\beta} - C_1\left(s^{\beta}{}_{;\alpha}^{;\alpha} + \frac{1+\eta}{1-\eta}\, s^{\alpha}{}_{;\alpha}^{;\beta}\right) = GC_1\left(\frac{\bar{p}^{\beta}}{{}_0\lambda\,{}_0\mu} - \frac{2\bar{\lambda}L}{1-\eta}\, w_{;\alpha}^{;\alpha\beta}\right). \tag{2}$$

These equations describe the infinitesimal deformation of an isotropic, sandwich plate with dissimilar facings and a weak, incompressible core. We shall eliminate the derivatives of the unknown functions $w$ and $s^{\beta}$ ($\beta = 1, 2$) which appear in (1) and (2) by using integral transformations with appropriate kernels.

Assume that the plate covers an area $S$ with a boundary $\Gamma$. Let $\phi_{(i)}(\boldsymbol{\theta})$ be orthogonal eigenfunctions corresponding to the following boundary value problems

$$\phi_{(i);\alpha}^{;\alpha} + (k_{(i)})^2\phi_{(i)} = 0 \text{ in } S \tag{3}$$

$$e_{(i)}\phi_{(i)} + f_{(i)}\phi_{(i),\alpha}n^{\alpha} = 0 \text{ on } \Gamma \tag{4}$$

(no summation with respect to $i$)

where $e_{(i)}, f_{(i)}$ are constants and $\mathbf{n}$ is the outward normal to $\Gamma$. Using the generalized Green's theorem[3],

$$\int_S F_{;\alpha}\,\mathrm{d}S = \int_{\Gamma} Fn_{\alpha}\,\mathrm{d}\Gamma \tag{5}$$

where $F$ is an arbitrary tensor field, the equation (A2) (see the Appendix) is easily derived.

Multiply the scalar equation (1) by the orthogonal functions $\phi_{(3)}$ and integrate the result over $S$. In order to present the transforms of the derivatives of $w$ in terms of the transform $w_{mn}$ of $w$, we define

$$w_{mn} \equiv \int_S w\phi_{(3)}\,\mathrm{d}S. \tag{6}$$

The boundary condition (4) and the transform (6) have been substituted into formula (A2) resulting in†

$$\int_S w_{;\alpha\beta}^{;\alpha\beta}\phi_{(3)}\,\mathrm{d}S = (k_{(3)})^4 w_{mn} + \int_\Gamma \{(k_{(3)})^2[(f_{(3)}/e_{(3)})w_{;}^{\alpha}n_\alpha + w] - [(f_{(3)}/e_{(3)})w_{;\alpha}^{;\alpha\beta}n_\beta + w_{;\alpha}^{;\alpha}]\}\phi_{(3);}^{\gamma}n_\gamma\,\mathrm{d}\Gamma, \tag{7}$$

$$\int_S w_{;\alpha\beta\gamma}^{;\alpha\beta\gamma}\phi_{(3)}\,\mathrm{d}S = -(k_{(3)})^6 w_{mn} + \int_\Gamma \{-(k_{(3)})^4[(f_{(3)}/e_{(3)})w_{;}^{\alpha}n_\alpha + w] + (k_{(3)})^2[(f_{(3)}/e_{(3)})w_{;\alpha}^{;\alpha\beta}n_\beta + w_{;\alpha\beta}^{;\alpha\beta}] - (f_{(3)}/e_{(3)})w_{;\alpha\beta}^{;\alpha\beta\gamma}n_\gamma - w_{;\alpha\beta}^{;\alpha\beta}\}\phi_{(3);}^{\rho}n_\rho\,\mathrm{d}\Gamma \tag{8}$$

when $e_{(3)} \neq 0$.

When $e_{(3)} = 0$ the following relations are valid

$$\int_S w_{;\alpha\beta}^{;\alpha\beta}\phi_{(3)}\,\mathrm{d}S = (k_{(3)})^4 w_{mn} - \int_\Gamma [(k_{(3)})^2 w_{;}^{\alpha} - w_{;\beta}^{;\beta\alpha}]n_\alpha\phi_{(3)}\,\mathrm{d}\Gamma, \tag{9}$$

$$\int_S w_{;\alpha\beta\gamma}^{;\alpha\beta\gamma}\phi_{(3)}\,\mathrm{d}S = -(k_{(3)})^6 w_{mn} + \int_\Gamma [(k_{(3)})^4 w_{;}^{\alpha} - (k_{(3)})^2 w_{;\beta}^{;\beta\alpha} + w_{;\beta\gamma}^{;\beta\gamma\alpha}]n_\alpha\phi_{(3)}\,\mathrm{d}\Gamma. \tag{10}$$

In order to eliminate the quantity $w_{;\alpha\beta\gamma}^{;\alpha\beta}(\Gamma)$ which does not have a direct physical meaning, the equation (40) from paper[1]

$$C_2\,w_{;\alpha\beta}^{;\alpha\beta} - p^3 - \frac{\bar{\lambda}}{2\lambda}s^{\alpha}{}_{;\alpha} - \tfrac{1}{2}c^{\alpha}{}_{;\alpha} = 0 \tag{11}$$

is taken into account. For the same reason, the term $w_{;\alpha\beta}^{;\alpha\beta\gamma}$ is eliminated by assuming $f_{(3)} = 0$. Hence $\phi_{(3)}$ must satisfy the boundary condition $\phi_{(3)}(\Gamma) = 0$. The transformed equation (1) now takes the form

$$-(k_{(3)})^4[(k_{(3)})^2 + A]w_{mn} + BP_{mn} = [(k_{(3)})^2 + A]\int_\Gamma [(k_{(3)})^2 w - w_{;\alpha}^{;\alpha}]\,\phi_{(3);}^{\beta}n_\beta\,\mathrm{d}\Gamma + \frac{1}{C_2}\int_\Gamma \left(\frac{\bar{\lambda}}{2\lambda}s^{\alpha}{}_{;\alpha} + \tfrac{1}{2}c^{\alpha}{}_{;\alpha} + p^3\right)\phi_{(3);}^{\beta}n_\beta\,\mathrm{d}\Gamma \tag{12}$$

where

$$P_{mn} \equiv \int_S P\phi_{(3)}\,\mathrm{d}S. \tag{13}$$

Equations (2) are transformed by multiplying them by the scalar function $\phi_{(i)}$ ($i = 1$ or $2$) and integrating over $S$. Denoting by

$$s_{(i)mn}^{\beta} \equiv \int_S s^{\beta}\phi_{(i)}\,\mathrm{d}S \tag{14}$$

† Note, that both $\phi_{(3)}$ and $k_{(3)}$ depend on an infinite sequence of indices ($m$, $n$ say) which have been omitted here for brevity.

the Fourier transforms of $s^\beta$, and using expressions (A3), (A4), and (A5) the following transformed equations are obtained

$$s^\beta_{(i)mn}[1 + C_2(k_{(i)})^2] - \frac{1+\eta}{1-\eta} C_1 \int_S s^\alpha \phi_{(i);\alpha}{}^{\beta} \, dS$$
$$+ \frac{2GC_1\bar{\lambda}L}{1-\eta}(k_{(i)})^2 \int_S w\phi_{(i);}{}^{\beta} \, dS = C_1 \int_\Gamma [\phi_{(i)} s^\beta{}_;{}^\alpha - \phi_{(i);}{}^\alpha s^\beta] n_\alpha \, d\Gamma$$
$$+ C_1 \frac{1+\eta}{1-\eta} \int_\Gamma [\phi_{(i)} s^\alpha{}_{;\alpha} n^\beta - \phi_{(i);}{}^\beta s^\alpha n_\alpha] \, d\Gamma$$
$$+ \frac{2GC_1\bar{\lambda}L}{1-\eta}(k_{(i)})^2 \int_\Gamma w\phi_{(i)} n^\beta \, d\Gamma - \frac{2GC_1\bar{\lambda}L}{1-\eta} \int_\Gamma [\phi_{(i)} w_;{}^{\beta\alpha}$$
$$- \phi_{(i);}{}^\alpha w_;{}^\beta] n_\alpha \, d\Gamma + \frac{GC_1}{{}_0\lambda\,{}_0\mu} \int_S \bar{p}\phi_{(i)} \, dS. \tag{15}$$

Since $w$ is known from (12) the main difficulty consists in investigation of the second integral on the left hand side of equation (15). This integral may be represented as a linear finite function of the transforms (14) only if $\phi_{(i)}$ are orthogonal to their derivatives $\phi_{(i);\alpha}{}^{\beta}$. Otherwise, the transformed equation (15) will have a form of an infinite system of linear algebraic equations. Since the orthogonality mentioned above takes place only in very few special cases a different procedure will be developed to transform equation (2).

## ALTERNATIVE TRANSFORMATION OF EQUATIONS (2)

Covariant differentiation of equations (2) with respect to $\theta^\gamma$ and subsequent contraction leads to

$$s^\beta{}_{;\beta} - \frac{2C_1}{1-\eta} s^\beta{}_{;\alpha\beta}{}^{\alpha} = GC_1\left(\frac{\bar{p}^\beta{}_{;\beta}}{{}_0\lambda\,{}_0\mu} - \frac{2\bar{\lambda}L}{1-\eta} w_{;\alpha\beta}{}^{\alpha\beta}\right). \tag{16}$$

Let $s_{mn}$ be the integral transform of $s^\beta{}_{;\beta}$ with regard to a system of orthogonal functions $\phi$ satisfying equations (3) and (4), viz.

$$s_{mn} \equiv \int_S s^\beta{}_{;\beta}\phi \, dS \tag{17}$$

The inverse transform of (17) is then

$$s^\beta{}_{;\beta} = \sum_{m,n} s_{mn}\phi/\|\phi\|^2 \tag{18}$$

where $\phi$, and its eigenvalues $k$, depend on the summation indices $m$, $n$.

Multiplying (16) by $\phi$ and integrating over $S$ leads to the following expression for $s_{mn}$

$$\frac{1-\eta+2C_1(k)^2}{1-\eta} s_{mn} = GC_1 \int_S \left(\frac{\bar{p}^\beta{}_{;\beta}}{{}_0\lambda\,{}_0\mu} - \frac{2\bar{\lambda}L}{1-\eta} w_{;\alpha\beta}{}^{\alpha\beta}\right)\phi \, dS$$
$$+ \frac{2C_1}{1-\eta}\int_\Gamma (s^\beta{}_{;\beta}{}^{\alpha}\phi - s^\beta{}_{;\beta}\phi_;{}^\alpha) n_\alpha \, d\Gamma. \tag{19}$$

Substituting (19) and (18) into equations (2) one obtains

$$s^{\beta} - C_1 s^{\beta}{}_{;\alpha}{}^{\alpha} = F^{\beta} \tag{20}$$

where

$$F^{\beta} = GC_1\left(\frac{\bar{p}^{\beta}}{{}_0\lambda\,{}_0\mu} - \frac{2\lambda L}{1-\eta} w_{;\alpha}{}^{\alpha\beta}\right)C_1(1+\eta)$$
$$\times\left\{\sum_{m,n}\frac{1}{1-\eta+2C_1(k)^2}\left[GC_1\int_S\left(\frac{\bar{p}_{;\gamma}{}^{\gamma}}{{}_0\lambda\,{}_0\mu} - \frac{2\lambda L}{1-\eta} w_{;\alpha\gamma}{}^{\alpha\gamma}\right)\phi\,\mathrm{d}S\right.\right.$$
$$\left.\left.+\frac{2C_1}{1-\eta}\int_\Gamma (s_{;\gamma}{}^{\gamma\alpha}\phi - s_{;\gamma}{}^{\gamma}\phi_{;}{}^{\alpha})n_\alpha\,\mathrm{d}\Gamma\right]/\|\phi\|^2\right\}_{;}{}^{\beta}. \tag{21}$$

Equations (20) can be now transformed in the same manner as equations (2) were in the previous section. Multiplying equation (20) by $\phi_{(i)}$, integrating over $S$, and applying the generalized Green's theorem yields

$$s^{\beta}_{(i)mn}[1 + C_1(k_{(i)})^2] = C_1\int_\Gamma[\phi_{(i)}s^{\beta}{}_{;}{}^{\alpha} - \phi_{(i);}{}^{\alpha}s^{\beta}]n_\alpha\,\mathrm{d}\Gamma + \int_S F^{\beta}\phi_{(i)}\,\mathrm{d}S. \tag{22}$$

In this way the inconvenient integral from equation (15) has been removed. It should be mentioned, however, that a new difficulty appears here: it is now necessary to differentiate an infinite series in equation (21).

## EXAMPLE: RECTANGULAR PLATE

A rectangular plate $a \times b$ will be investigated as the simplest possible case.

Assume:

$$\begin{aligned}\phi_{(1)} &= \cos(m\pi\theta^1)\sin(n\pi a\theta^2/b),\\ \phi_{(2)} &= \sin(m\pi\theta^1)\cos(n\pi a\theta^2/b),\\ \phi_{(3)} &= \sin(m\pi\theta^1)\sin(n\pi a\theta^2/b),\end{aligned} \tag{23}$$

where $0 \le \theta^1 \le 1$, $0 \le \theta^2 \le b/a$. Let the characteristic dimension $L$ be equal to $a$.

The eigenvalues corresponding to $\phi_{(i)}$ are

$$(k_{(i)})^2 = \pi^2(m^2 + n^2a^2/b^2) \equiv k_{mn}{}^2 \qquad (i = 1, 2, 3). \tag{24}$$

Denoting

$$\Omega(\theta_1, \theta_2) \equiv (k_{mn}{}^2 + A)(k_{mn}{}^2 + w - w_{;\alpha}{}^{\alpha}) + \frac{1}{C_2}\left\{\frac{\lambda}{2\lambda}s^{\alpha}{}_{;\alpha} + \tfrac{1}{2}c^{\alpha}{}_{;\alpha} + p^3\right\} \tag{25}$$

and utilizing (23) one can represent equation (12) in the following form

$$-k_{mn}{}^4(k_{mn}{}^2 + A)w_{mn} + PB_{mn} = \int_0^1\Big[(-1)^n\Omega(\theta^1, b/a) - \Omega(\theta^1, 0)\Big]\frac{n\pi a}{b}\sin(m\pi\theta^1)\,\mathrm{d}\theta^1$$
$$+\int_0^{b/a}\Big[(-1)^m\Omega(1, \theta^2) - \Omega(0, \theta^2)\Big]m\pi\sin\left(\frac{n\pi a}{b}\theta^2\right)\mathrm{d}\theta^2, \tag{26}$$

where

$$w_{mn} = a^2 \int_0^1 \int_0^{b/a} w\phi_{(3)}\, d\theta^1\, d\theta^2. \tag{27}$$

Assuming

$$s^1{}_{mn} = a^2 \int_0^1 \int_0^{b/a} s^1\phi_{(1)}\, d\theta^1\, d\theta^2,$$

$$s^2{}_{mn} = a^2 \int_0^1 \int_0^{b/a} s^2\phi_{(2)}\, d\theta^1\, d\theta^2, \tag{28}$$

one can represent the left hand sides of equations (15) in the following forms

$$(1 + C_1 k_{mn}{}^2)s^1{}_{mn} + \frac{1+\eta}{1-\eta}\, C_1 m\pi^2[ms^1{}_{mn} + n(a/b)s^2{}_{mn}] - \frac{2GC_1\bar{\lambda}a}{1-\eta}\, k_{mn}{}^2 m\pi w_{mn} = D_{1mn} \tag{29}$$

and

$$(1 + C_1 k_{mn}{}^2)s^2{}_{mn} + \frac{1+\eta}{1-\eta}\, C_1 n\pi^2(a/b)[ms^1{}_{mn} + n(a/b)s^2{}_{mn}]$$

$$- \frac{2GC_1\bar{\lambda}a}{1-\eta}\, k_{mn}{}^2 n\pi(a/b)w_{mn} = D_{2mn} \tag{30}$$

These results were obtained by assuming $i = 1$ when $\beta = 1$, and $i = 2$ when $\beta = 2$.

The following abbreviations have been introduced here

$$D_{1mn} \equiv C_1 \int_0^{b/a} [(-1)^m X_1(1, \theta^2) - X_1(0, \theta^2)]\sin\left(\frac{n\pi a}{b}\,\theta^2\right) d\theta^2$$

$$- \frac{n\pi a}{b}\, C_1 \int_0^1 [(-1)^n X_2(\theta^1, b/a) - X_2(\theta^1, 0)]\cos(m\pi\theta^1)\, d\theta^1$$

$$+ \frac{GC_1}{{}_0\lambda\, {}_0\mu} \int_0^1 \int_0^{b/a} \bar{p}\phi_{(1)}\, d\theta^1\, d\theta^2 \tag{31}$$

$$D_{2mn} \equiv C_1 \int_0^1 [(-1)^n X_3(\theta^1, b/a) - X_3(\theta^1, 0)]\sin(m\pi\theta^1)\, d\theta^1$$

$$- m\pi C_1 \int_0^{b/a} [(-1)^m X_4(1, \theta^2) - X_4(0, \theta^2)]\cos\left(\frac{n\pi a}{b}\,\theta^2\right) d\theta^2$$

$$+ \frac{GC_1}{{}_0\lambda\, {}_0\mu} \int_0^1 \int_0^{b/a} \bar{p}\phi_{(2)}\, d\theta^1\, d\theta^2 \tag{32}$$

where

$$X_1(\theta^1, \theta^2) = s^1{}_{,1} + \frac{1+\eta}{1-\eta}\,(s^1{}_{,1} + s^2{}_{,2}) + \frac{2G\bar{\lambda}a}{1-\eta}\,(k_{mn}{}^2 w - w_{,11}),$$

$$X_2(\theta^1, \theta^2) = s^1 - \frac{2G\bar{\lambda}a}{1-\eta}\, w_{,1},$$

$$X_3(\theta^1, \theta^2) = s^2{}_{,2} + \frac{1+\eta}{1-\eta}(s^1{}_{,1} + s^2{}_{,2}) + \frac{2G\bar{\lambda}a}{1-\eta}(k_{mn}{}^2 w - w_{,11}),$$

$$X_4(\theta^1, \theta^2) = s^2 - \frac{2G\bar{\lambda}a}{1-\eta} w_{,2}.$$

Solving the system of algebraic equations (29) and (30) the following expressions for the Fourier transforms $s^1{}_{mn}$, and $s^2{}_{mn}$ are obtained:

$$s^1{}_{mn} = \frac{1}{\Delta_{mn}}\{(1 + C_1 k_{mn}{}^2)[(1-\eta)D_{1mn} + 2GC_1\bar{\lambda}am\pi k_{mn}{}^2 w_{mn}]$$
$$+ (1+\eta)C_1 n\pi^2(a/b)[nD_{1mn}(a/b) - mD_{2mn}]\}, \tag{34}$$

$$s^2{}_{mn} = \frac{1}{\Delta_{mn}}\{(1 + C_1 k_{mn}{}^2)[(1-\eta)D_{2mn} + 2GC_1\bar{\lambda}an\pi(a/b)k_{mn}{}^2 w_{mn}]$$
$$+ (1+\eta)C_1 m\pi^2[mD_{2mn} - nD_{1mn}(a/b)]\}, \tag{35}$$

where $w_{mn}$ is the solution of the algebraic equation (26), and

$$\Delta_{mn} = 1 - \eta + (3-\eta)C_1 k_{mn}{}^2 + 2C_1{}^2 k_{mn}{}^4. \tag{36}$$

Once the Fourier transforms $w_{mn}$, $s^1{}_{mn}$, $s^2{}_{mn}$ are known the functions $w$, $s^1$, and $s^2$ are obtained from the inversion formulae:

$$w = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}\phi_{(3)},$$

$$s^1 = \frac{4}{ab}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(1 - \tfrac{1}{2}\delta_{0m})s^1{}_{mn}\phi_{(1)},$$

$$s^2 = \frac{4}{ab}\sum_{m=n}^{\infty}\sum_{n=0}^{\infty}(1 - \tfrac{1}{2}\delta_{0n})s^2{}_{mn}\phi_{(2)}, \tag{37}$$

where $\delta_{0m}$, $\delta_{0n}$, is Kronecker's delta.

Notice that in general the expressions for Fournier transforms depend on the integrals of the boundary values of the unknown functions and/or their derivatives. These integrals vanish in special cases only. Otherwise it is necessary to use available boundary conditions—and this requires differentiation of Fourier series—to eliminate these unknown quantities.

*Acknowledgement*—The author gratefully acknowledges stimulating discussion with Dr. Ronald Brand from the Mechanical Engineering Department of the University of Connecticut.

## REFERENCES

1. G. A. Wempner and J. L. Baylor, General theory of sandwich plates with dissimilar facings. *Int. J. Solids Struct.* **1**, 157(1965).
2. R. Solecki, Some invariant expressions for integral transforms. Submitted for publication.
3. J. Serrin, Mathematical principles of classical fluid mechanics. *Handbuch der Physik*, Vol. VIII/1. Springer (1959).

## APPENDIX

The generalized Green's theorem

$$\int_S F_{;\alpha}\,\mathrm{d}S = \int_\Gamma F n_\alpha\,\mathrm{d}\Gamma \tag{A1}$$

can be used to generate several useful transformation formulae. Assuming, for instance, $F = \phi\Psi_{;}{}^{\alpha} - \phi_{;}{}^{\alpha}\Psi$ where $\phi$ and $\Psi$ are scalar fields one gets the well known formula

$$\int_S (\phi\Psi_{;\alpha}{}^{\alpha} - \phi_{;\alpha}{}^{\alpha}\Psi)\,\mathrm{d}S = \int_\Gamma (\phi\Psi_{;}{}^{\alpha} - \phi_{;}{}^{\alpha}\Psi)n_\alpha\,\mathrm{d}\Gamma. \tag{A2}$$

Similarly, $F = \phi u^{\beta}{}_{;}{}^{\alpha} - \phi_{;}{}^{\alpha}u^{\beta}$ yields

$$\int_S (\phi u^{\beta}{}_{;\alpha}{}^{\alpha} - \phi_{;\alpha}{}^{\alpha}u^{\beta})\,\mathrm{d}S = \int_\Gamma (\phi u^{\beta}{}_{;}{}^{\alpha} - \phi_{;}{}^{\alpha}u^{\beta})n_\alpha\,\mathrm{d}\Gamma \tag{A3}$$

where $\phi$ is a scalar and $\boldsymbol{u}$ a vector.
If $F = \phi u^{\beta}{}_{;\beta}$, the expression

$$\int_S \phi u^{\beta}_{;\beta\alpha}\,\mathrm{d}S = -\int_S \phi_{;\alpha} u^{\beta}{}_{;\beta}\,\mathrm{d}S + \int_\Gamma \phi u^{\beta}{}_{;\beta} n_\alpha\,\mathrm{d}\Gamma$$

results. In order to transform the first integral on the right hand side the substitution $F = \phi_{;\beta} u^{\alpha}$ is used in (A1) resulting in

$$\int_S \phi_{;\beta} u^{\alpha}{}_{;\alpha}\,\mathrm{d}S = -\int_S \phi_{;\beta\alpha} u^{\alpha}\,\mathrm{d}S + \int_\Gamma \phi_{;\beta} u^{\alpha} n_\alpha\,\mathrm{d}\Gamma$$

so that

$$\int_S \phi u^{\beta}_{;\beta\alpha}\,\mathrm{d}S = \int_S u^{\beta}\phi_{;\alpha\beta}\,\mathrm{d}S + \int_\Gamma [\phi u^{\beta}{}_{;\beta} n_\alpha + \phi_{;\alpha} u^{\beta} n_\beta]\,\mathrm{d}\Gamma. \tag{A4}$$

If $F = \phi\Psi$ one obtains

$$\int_S \phi_{;\alpha}\Psi\,\mathrm{d}S = -\int_S \phi\Psi_{;\alpha}\,\mathrm{d}S + \int_\Gamma \phi\Psi n_\alpha\,\mathrm{d}\Gamma. \tag{A5}$$

**Абстракт**—Даются интегральные преобразования производных функций, появляющихся в уравнениях изгиба многослойных пластин, в виде преобразований этих функций и соответствующих граничных значений. Применение обобщенной формулы. Грина дает возможность представить эти преобразования в инвариантной форме. В качестве примера, приводится расчет прямоугольной пластинки.